Doctoral School of Physics - Eötvös Loránd University (ELTE)

Four-semester report

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Ph.D. Thesis topic:

Study of solitons and applications with analytical and numerical methods

1) *Introduction and Description of research work carried out:*

Historically, in 1837 the British Association for the Advancement of Science appointed John Scott Russell and Sir John Robison to a "Committee on Waves" charged with conducting laboratory experiments to investigate water-wave phenomena [1, 2]. They published a substantial report in which surface waves were classified into four types: solitary, oscillatory (periodic wave trains), capillary (due to surface tension effects), and corpuscular (compressive sound waves propagating through water). After their experimental study, various mathematicians and physicists investigated the topic. In later years, Boussinesq, Rayleigh, and McCowan came up with theories to describe solitary waves on an inviscid and incompressible fluid. In particular, Boussinesq [2] showed in 1871 that suitable approximations to the governing equations lead, for sufficiently small amplitudes, to solitary wave solutions of the type described by Scott Russell in [1]. By balancing nonlinearity and dispersion, Korteweg and de Vries [3] derived using a systematic approximation procedure applied to the governing equations for water waves a simple nonlinear model equation for long waves that admits solitary-wave solutions. The KdV theory is a first-order approximation that adequately explains the wave properties observed by Scott Russell. Since the 1950s, the KdV equation and other equations that admit solitary wave solutions have been the subject of intense studies. Also, numerous experiments investigating head-on solitary wave, their behavior, and collisions of solitary waves were carried out [1, 2, 4].

This report covers studies on the dissipation relation in the presence of viscosity and the approximation Navier-Stokes (NS) equations by finding Taylor coefficients of a velocity field. In this way, we can construct a velocity model that is polynomial in terms of the distance from the point of expansion.

I) *Damping of surface waves on finite depth viscous fluids*

To describe the nonlinear wave phenomenon remarkably in the ocean, various evolution equations have been proposed and studied. In shallow water, the Boussinesq equations for two-dimensional waves, the Korteweg-de Vries (KdV) equation for uni-directional waves, the Kadomtsev-Petviashvili (KP) equation for weakly two-dimensional waves and their solitary wave solutions are very well-known and of interests to all disciplines [5]. Many studies have attempted to assess the significance of dissipation. The viscous effects are a problematic situation as there is always a degree of ambiguity surrounding the concept of dissipation. The effect of viscosity on free oscillatory waves on deep water was studied by Boussinesq and Lamb, among others. Basset also worked on viscous damping of water waves [5, 6].

The consciousness of the effect of viscous dissipation is required in the assessment of the various process of wave generation [7]. Followed by previous studies we consider linear surface waves on a viscous, incompressible fluid of finite depth h. The coordinates x and y are horizontal, z is vertical. The origin lies at the undisturbed fluid surface. Suppose that the flow corresponding to the surface wave does not depend on y. We start with the Ansatz

$$u(x,z,t) = f'(z)e^{i(kx-\omega t)} \tag{1}$$

$$w(x,z,t) = -ik\, f(z)e^{i(kx-\omega t)} \tag{2}$$

for the non-vanishing velocity components. Note that ω is in general complex, its imaginary part describing the damping and the Ansatz automatically satisfies the incompressibility condition. The linearized Navier-Stokes equation may be written as

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\, \Delta \mathbf{v} = \nabla\left(\frac{-1}{\rho} P(x,z,t) - gz\right) \tag{3}$$

So

$$\begin{aligned}
\frac{\partial u}{\partial t} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) &= e^{i(kx-\omega t)}\left[\left(k^2\nu - i\omega\right) f' - \nu f'''\right] = \frac{\partial}{\partial x}\left(-\frac{1}{\rho}P - gz\right) \\
\frac{\partial w}{\partial t} - \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) &= ik\, e^{i(kx-\omega t)}\left[\left(-k^2\nu + i\omega\right) f + \nu f''\right] = \frac{\partial}{\partial z}\left(-\frac{1}{\rho}P - gz\right)
\end{aligned} \tag{4}$$

Which leads to a differential equation for $f(z)$

$$f'''' + \left(i\frac{\omega}{\nu} - 2k^2\right) f'' - \left(i\frac{\omega}{\nu} - k^2\right)k^2 f = 0 \tag{5}$$

The Eq (5) has exponential solutions $f = \exp(\kappa z)$. For the exponent κ, we get

$$\kappa^4 + \left(i\frac{\omega}{\nu} - 2k^2\right)\kappa^2 - \left(i\frac{\omega}{\nu} - k^2\right)k^2 f = 0 \tag{6}$$

The solutions are

$$\kappa_{1,2} = \pm k \tag{7}$$

$$\kappa_{3,4} = \pm\sqrt{k^2 - i\frac{\omega}{\nu}} \tag{8}$$

For brevity, we shall use the notation κ for $\kappa_3 = -\kappa_4$ and k for $\kappa_1 = -\kappa_2$. The general solution f may be given as

$$f = a_1 \cosh\left[k(z+h)\right] + a_2 \sinh\left[k(z+h)\right] + b_1 \cosh\left[\kappa(z+h)\right] + b_2 \sinh\left[\kappa(z+h)\right] \tag{9}$$

Where a_1, a_2, b_1 and b_2 are integration constants and h stands for the fluid depth. Then the boundary conditions at the bottom,

$$u(z=-h) = w(z=-h) = 0 \tag{10}$$

Imply that

$$a_1 + b_1 = 0 \tag{11}$$

$$a_2 k + b_2 \kappa = 0 \tag{12}$$

$$a_1 = A, \quad b_2 = B, \quad b_1 = -A, \quad a_2 = -\frac{\kappa}{k} B \tag{13}$$

Which is expressed in terms of the new constants A and B. Hence for f we get

$$f = A\cosh\left[k(z+h)\right] - \frac{\kappa}{k} B\sinh\left[k(z+h)\right] - A\cosh\left[\kappa(z+h)\right] + B\sinh\left[\kappa(z+h)\right] \tag{14}$$

Upon integrating the x component of the Navier-Stokes equation with respect to x, we get the pressure as

$$p = p_0 - \rho g z - \rho\, \mathrm{e}^{(i\varphi)}\left(-\frac{\omega}{k} f' - i\nu k f' + i\frac{\nu}{k} f'''\right) \tag{15}$$

Where $\varphi = kx - \omega t$. At the fluid surface we have the boundary conditions that the strain forces are continuous, therefore (in linear approximation) we have

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \tag{16}$$

for the shear and

$$p - 2\rho\nu \frac{\partial w}{\partial z} = p_0 \tag{17}$$

for the pressure. Eq (16) implies

$$f'' + k^2 f = 0 \qquad (\text{at } z = 0) \tag{18}$$

While the Eq (17) implies

$$-g\eta - \mathrm{e}^{(i\varphi)}\left(-\frac{\omega}{k} f' - i\nu k\, f' + i\frac{\nu}{k} f'''\right) + 2i\nu k\, f' \mathrm{e}^{(i\varphi)} = 0 \tag{19}$$

Note that here $z = \eta(x,t)$ stands for the deviation of the fluid surface from equilibrium. We have at the surface (again in linear approximation)

$$\frac{\partial \eta}{\partial t} = w \tag{20}$$

Note that on the right-hand side, we may set $z = 0$. Putting here the expression of v_z (i,e., Eq (2)) and that of η from Eq (19), we have

$$k^2 f - \frac{1}{g}\left(\omega^2 + 3i\omega\nu k^2\right) f' + i\frac{\omega\nu}{g} f''' = 0 \tag{21}$$

Alternatively, in terms of κ

$$k^2 f + \frac{\nu^2}{g}(\kappa^2 - k^2)(\kappa^2 + 2k^2)\, f' - \frac{\nu^2}{g}(\kappa^2 - k^2)\, f''' = 0 \tag{22}$$

Here again $z=0$. Inserting now the solution(14) into Eqs (18) and (22) we obtain a linear homogeneous system of equation for the coefficients A and B. Furthermore, the vanishing of the determinant of this system (which is the condition for the existence of a nontrivial solution) may be expressed in terms of the dimensionless variables

$$K = kh \tag{23}$$

$$Q = \kappa h = \sqrt{1 - i\frac{\omega}{\nu k^2}}\; kh \tag{24}$$

and

$$p = \frac{\nu^2}{gh^3} \tag{25}$$

as

$$\begin{gathered} K\left(Q\sinh K\cosh Q - K\cosh K\sinh Q\right) + p\Big[-4K^2Q\left(K^2+Q^2\right) \\ +Q\left(Q^4+2K^2Q^2+5K^4\right)\cosh K\cosh Q - K\left(Q^4+6K^2Q^2+K^4\right)\sinh K\sinh Q\Big]=0 \end{gathered} \tag{26}$$

Now by introducing ε, one can rewrite (26) in terms of $\frac{Q}{K}$.

$$\varepsilon = \left(\frac{4\nu^2k^3}{g}\right)^{\frac{1}{4}} = \left(4pK^3\right)^{\frac{1}{4}} \tag{27}$$

So

$$\begin{gathered} \left(\frac{Q}{K}\tanh K - \tanh Q\right)\cosh K\,\cosh Q + \frac{\varepsilon^4}{4}\Bigg[-4\frac{Q}{K}\left(1+Q^2\right) \\ +\frac{Q}{K}\left(\left(\frac{Q}{K}\right)^4 + 2\left(\frac{Q}{K}\right)^2 + 5\right)\cosh K\cosh Q - \left(\left(\frac{Q}{K}\right)^4 + 6\left(\frac{Q}{K}\right)^2 + 1\right)\sinh K\sinh Q\Bigg] = 0 \end{gathered} \tag{28}$$

Note that if Q is a solution of the Eq (28), so is $-Q$. On the other hand, this sign does not matter when calculating ω. Henceforth we assume that the real part of Q is positive, and thus $\tanh Q \to 1$ when $|Q| \to \infty$. We can also rewrite Q in terms of K, ε and $S = \frac{\omega}{\sqrt{gk}}$.

$$2K^2S = i\varepsilon^2\left(Q^2 - K^2\right) \tag{29}$$

It is clear that in liquids of vanishing viscosity, as $\varepsilon \to 0$, S and K assume their inviscid significance and remain finite. Consequently, from (29), as $\varepsilon \to 0$ then $Q \to \infty$. Therefore, in the leading order Eq(28) reduces to

$$-4\left[\frac{Q}{K}\tanh(K) - 1\right] = \varepsilon^4\left[\left(\frac{Q}{K}\right)^5 + 2\left(\frac{Q}{K}\right)^3 + 5\frac{Q}{K} - \tanh(K)\left(\left(\frac{Q}{K}\right)^4 + 6\left(\frac{Q}{K}\right)^2 + 1\right)\right] \tag{30}$$

And

$$K\tanh K + pQ_0^4 = 0 \tag{31}$$

$$Q_0^2 = -i\sqrt{\frac{K\tanh K}{p}} \tag{32}$$

Here the negative sign has been chosen in order to get positive real part of angular frequency via Eq (29). Further, according to the convention mentioned above, we have

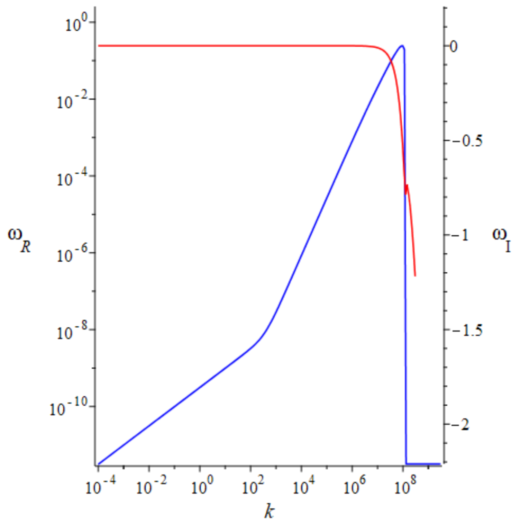

Figure 1. The dispersion relation for gravity waves propagating at the interface of air and water in the presence of kinematic viscosity. Numerical solution of the Eq (34). The blue and the red lines show $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ respectively.

$$Q_0 = \frac{1-i}{\sqrt{2}}\left(\frac{K\tanh K}{p}\right)^{1/4} \tag{33}$$

Our results fully agree with the deep-water solutions, and particularly they lead to Zakharov's dispersion relation for small viscosity. The most significant benefit of (30) is that one can compute appropriate relation for finite depth and study shallow water $(K \to 0)$. In the limit of infinite depth, the dispersion relation can be shown to be the solution $\omega(k)$ of the following equation

$$\left(2 - i\frac{\omega}{\nu k^2}\right)^2 + \frac{g}{\nu^2 k^3} = 4\left(1 - i\frac{\omega}{\nu k^2}\right)^{\frac{1}{2}} \tag{34}$$

The physical solution is shown in Figure 1. for a wave propagating in the presence of $\nu = 1\times 10^{-6}\,\frac{m^2}{s}$. Furthermore, we seek a solution to the Eq (30) of the form

$$\frac{Q}{K} = q_o\varepsilon^{-1} + q_1 + q_2\varepsilon^1 + q_3\varepsilon^2 + \cdots \tag{35}$$

By substituting (35) into (30) and equating coefficients of powers of ε lead to the q_i

$$q_o = (1-i)\tanh^{\frac{1}{4}}(K) \tag{36}$$

$$q_1 = -\frac{1}{2}\operatorname{cosech}(2K) \tag{37}$$

$$q_2 = -(1-i)\coth^{\frac{1}{4}}(K)\frac{(Y+1)(Y+5)}{4Y(Y+4)} \tag{38}$$

$$q_3 = -i\tanh^{\frac{1}{2}}(K)\frac{Y^3 + 2Y^2 - 10Y - 10}{2Y^2(Y+4)} \tag{39}$$

Where $Y = 4\sinh^2(K)$.

According to the Eq (29), one can also express ω in terms of the powers of ε

$$S=\frac{\omega}{\sqrt{gh}}=s_o+s_1\varepsilon+s_2\varepsilon^2+s_3\varepsilon^3+\cdots \tag{40}$$

Therefore

$$s_o=\tanh^{\frac{1}{2}}(K) \tag{41}$$

$$s_1=-\frac{1}{2}(1+i)\tanh^{\frac{1}{4}}(K)\operatorname{cosech}(2K) \tag{42}$$

$$s_2=-i\frac{Y^2+5Y+2}{Y(Y+4)} \tag{43}$$

$$s_3=(1-i)\tanh^{\frac{3}{4}}(K)\frac{2Y^3+3Y^2-26Y-25}{4Y^2(Y+4)} \tag{44}$$

To next order,

$$Q_1K\tanh K-K^2+p\left(5Q_0^4Q_1-Q_0^4K\tanh K+2Q_0^3K^2\right)=0 \tag{45}$$

The last term in the bracket is higher order (significantly smaller) than the middle one, yet it must be kept because in case of deep fluid ($K\gg 1$) the middle term $-pQ_0^4K\tanh K$ is just canceled by the term $-K^2$ (second term in the Eq (45)).
Eq (45) yields

$$Q_1=-\frac{K}{2\sinh(2K)}-\frac{K^2}{2Q_0} \tag{46}$$

To this order, we have for the angular frequency

$$\begin{aligned}\omega &=\frac{\nu}{h^2}\sqrt{\frac{K\tanh(K)}{p}}-2i\nu k^2-i\frac{Q_0}{K\sinh(2K)}\nu k^2\\ &=\left[\sqrt{gk\tanh(kh)}-\frac{\sqrt{2\nu k^2\sqrt{gk\tanh(kh)}}}{2\sinh(2kh)}\right]-i\left[\frac{\sqrt{2\nu k^2\sqrt{gk\tanh(kh)}}}{2\sinh(2kh)}+2\nu k^2\right]\end{aligned} \tag{47}$$

The first term of the real part is the well-known dispersion relation of surface waves in ideal fluids [5]. As for damping, the leading term is the first one in the second bracket, proportional to $\sqrt{\nu}$, except in deep fluid. In deep fluid ($kh\to\infty$), this term vanishes. So one gets the well know damping exponent $2\nu k^2$.

II) No waves exist in shallow fluid layers

a. Gravity waves

Viscosity not only damps waves, but it can even prevent their propagation. This conclusion may be drawn from the Eq (28). In order to get a propagating wave with wave number K it is necessary and sufficient that Q has both a real and an imaginary part (see Eq (29)). By studying numerically the solution of the Eq(28) , we concluded that it is possible in the parameter domain under the curve shown in Fig. 2.
As such it can be concluded that no gravity waves can propagate if $p>0.085$[1]. Even if $p<0.085$, neither very long nor very short waves can propagate.

[1] In case of water this translates to $h<0.1\ mm$. for glycerin to $h<7.7\ mm$.

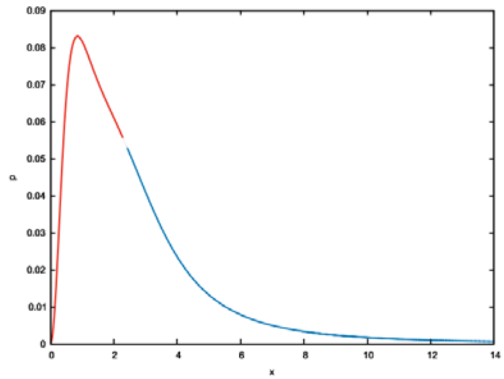

Figure 2. The maximal parameter p versus the scaled wave number $x = K$.
For parameters below the curve wave propagation is possible.

b. Capillary waves

Indeed, in such shallow fluid layers surface tension may not be negligible. If we take surface tension into account, we get instead of Eq (28)

$$K\left(Q\sinh K\cosh Q - K\cosh K\sinh Q\right)(1+sK^2) + p\Big[-4K^2Q\left(K^2+Q^2\right) + Q\left(Q^4+2K^2Q^2+5K^4\right)\cosh K\cosh Q - K\left(Q^4+6K^2Q^2+K^4\right)\sinh K\sinh Q\Big] = 0 \quad (48)$$

Where

$$s = \frac{\sigma}{\rho g h^2} \quad (49)$$

The solution of the Eq (48) yields a similar curve than Fig 2. If one plots the maxima of these curves versus s , one obtains Fig 3. As shown in Fig. 4, the curve starts almost like a constant. For large s values, we have a linear dependence, namely $p \approx 0.384 \times s$. This implies that wave propagation is not possible if

$$h < 2.6 \times \frac{\rho \nu^2}{\sigma} \quad (50)$$

The Eq (50) gives the limit for water $h < 3.56 \times 10^{-8}\ m$. As for glycerin, its parameters correspond to the beginning of the curve (Fig. 4) at $s = 0.086$ so here, the limit obtained for gravity waves is not significantly changed.
In summary, we predict that no waves can propagate in a thin liquid layer. This result may be checked in case of glycerin, where this critical layer thickness is $7.7\ mm$.

III) Fully Non-linear Viscous equations

We have aimed to apply a strategy similar to that applicable in case of ideal fluids to derive the KdV equation. Indeed, if it is sufficient to consider the first few terms of the Taylor expansion in terms of the vertical coordinate z , one may derive evolution equations for them from the NS equations. However, it turned out that this strategy was not applicable since it never happens that both K and Q are small, which would be a prerequisite. Indeed, if the modulus Q is plotted on the parameter plane, we get Fig. 5.

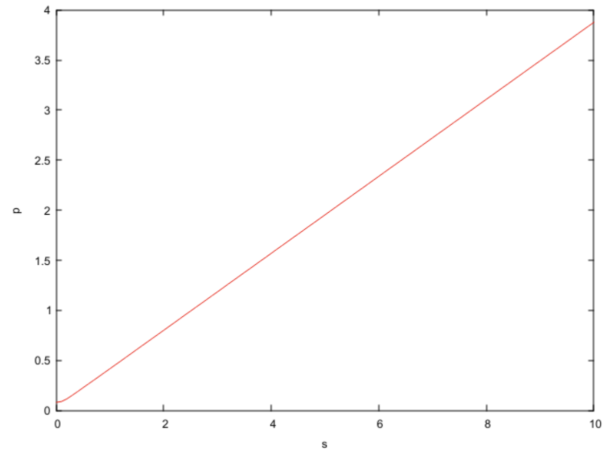

Figure 3. The maximal parameter *p* versus the scaled surface tension *s*. For parameters below the curve wave propagation is possible.

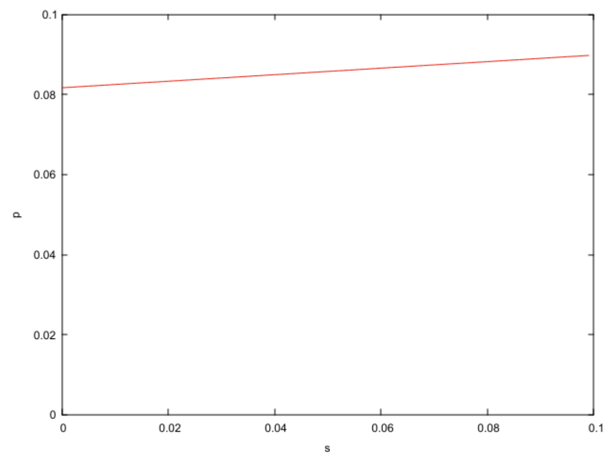

Figure 4. The maximal parameter *p* versus the scaled surface tension *s*. For parameters below the curve wave propagation is possible.

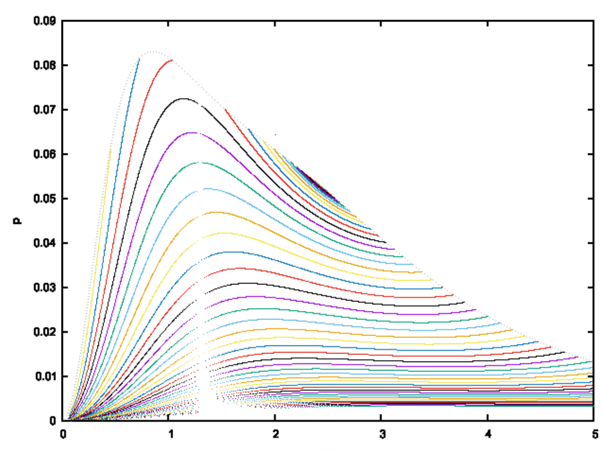

Figure 5. Level heights of modulus of *Q* in case of zero surface tension.

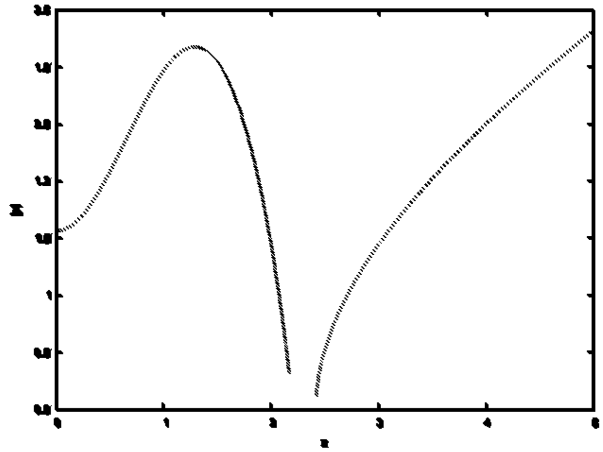

Figure 6. Modulus of Q along the critical curve (notations: x=K, y=Q).

One can see that there is a ridge in the middle of the picture (roughly vertically). It is now a question of how it behaves near the lower left corner. To this end, we plotted the modulus of Q along the critical $p(K)$ curve. The result is shown in Fig 6. This plot illustrates that where K is small, Q remains finite (larger than 1.5). Hence, the handling of nonlinear corrections needs another approach.

a. Taylor Expansion

It is also clear that the viscous terms can be added in the nonlinear equations. Here, we derive a set of nonlinear coupled partial differential equations that describe the surface wave as well as Taylor coefficients of velocity field [8].

$$u(x,z,t)=\frac{1}{\rho\nu}\left[\tau(x,t)z+\frac{1}{2}\gamma(x,t)z^2+\frac{1}{6}\sigma(x,t)z^3+\frac{1}{24}\lambda(x,t)z^4+\frac{1}{120}\mu(x,t)z^5+O\left(z^6\right)\right] \quad (51)$$

Where

$$\tau(x,t)=\rho\nu\frac{\partial u}{\partial z}(x,0,t) \quad (52)$$

$$\gamma(x,t)=\rho\nu\frac{\partial^2 u}{\partial z^2}(x,0,t)=\frac{\partial p}{\partial x}(x,0,t) \quad (53)$$

$$\sigma(x,t)=\rho\nu\frac{\partial^3 u}{\partial z^3}(x,0,t)$$

$$\lambda(x,t)=\rho\nu\frac{\partial^4 u}{\partial z^4}(x,0,t) \quad (54)$$

$$\mu(x,t)=\rho\nu\frac{\partial^5 u}{\partial z^5}(x,0,t)$$

Applying the boundary conditions, we will find the following equations

$$w+w_z\eta=\eta_t+u\eta_x \quad (55)$$

$$w_x+u_z+u_{zz}\eta-u_{xx}\eta=4u_x\eta_x \quad (56)$$

$$u_t + u_{tz}\eta + uu_x + wu_z + \eta w_t - \nu\left(u_{xx} + u_{zz} + \eta u_{zxx} + \eta u_{zzz} + \eta_x w_{xx} - \eta_x u_{zx}\right) = 2\nu\left(u_z + w_x\right)\eta_{xx} + 2\nu\left(\eta_x\right)\left(u_{zx} + w_{xx}\right) \tag{57}$$

$$\tau_t = 2\nu\tau_{xx} + \nu\sigma \tag{58}$$

$$\theta_t = 2\nu\theta_{xx} + \nu\lambda - \frac{1}{\rho\nu}\tau\tau_x \tag{59}$$

$$\sigma_t = \nu\sigma_{xx} - \nu\tau_{xxxx} + \nu\varepsilon - 2\frac{1}{\rho\nu}\tau\theta_x \tag{60}$$

Where $\eta = \eta(x,t)$ denotes the surface wave.

IV) Simulations

Simulation is another part of the research in which wave generation and wave dynamics will be studied. In order to simulate the problem, OpenFOAM software is used.

OpenFOAM is free, offering users the freedom to run, copy, distribute, study, change, and improve the software. Although many engineers handle computations on several commercial software, technologically OpenFoam is equivalent to commercial software. It is also able to create individualized solutions and offers an excellent scope of custom development. On the other hand, OpenFOAM is gaining considerable popularity in academic research and among industrial users, both as a research platform and a black-box CFD and structural analysis solver. The main ingredients of its design are

- Expressive and versatile syntax, allowing easy implementation of the complex physical model
- Extensive capabilities, including wealth of physical modeling, accurate and robust discretization, and sophisticated geometry handling, to the level present in commercial CFD
- Open architecture and open source development, where the complete source code is available to all users for customization and extension at no cost

In this step, the issue is the mechanism of generating solitary waves. To this end, there are different scenarios to generate solitary waves and make (un)even bottom. To tackle this problem, we finally install OlaFlow library. Fig 7 illustrates the simple boxes designed in OpenFoam. We also generate solitary waves and test their motion both in the same direction and in the opposite direction. Generation of two solitary waves and their interaction is now available in OlaFlow library as a simple example of this library.

Figure 7. Simple uneven bottom structure.

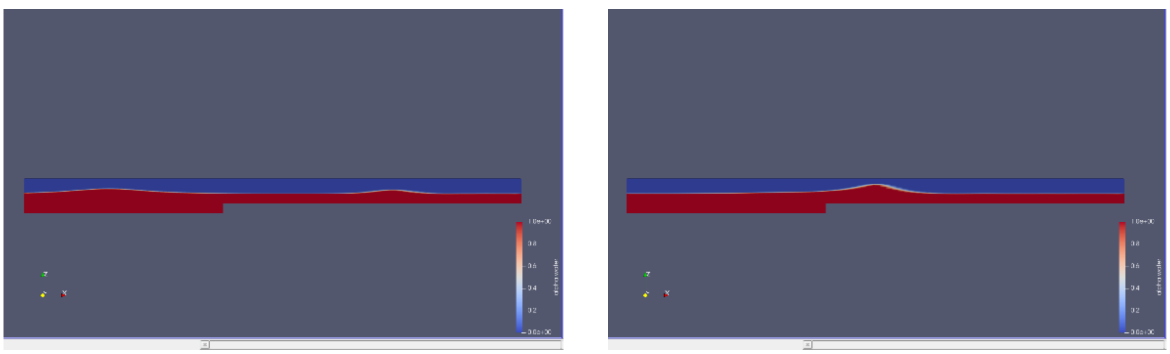

Figure 8. Interaction of two solitary waves that move in the opposite direction

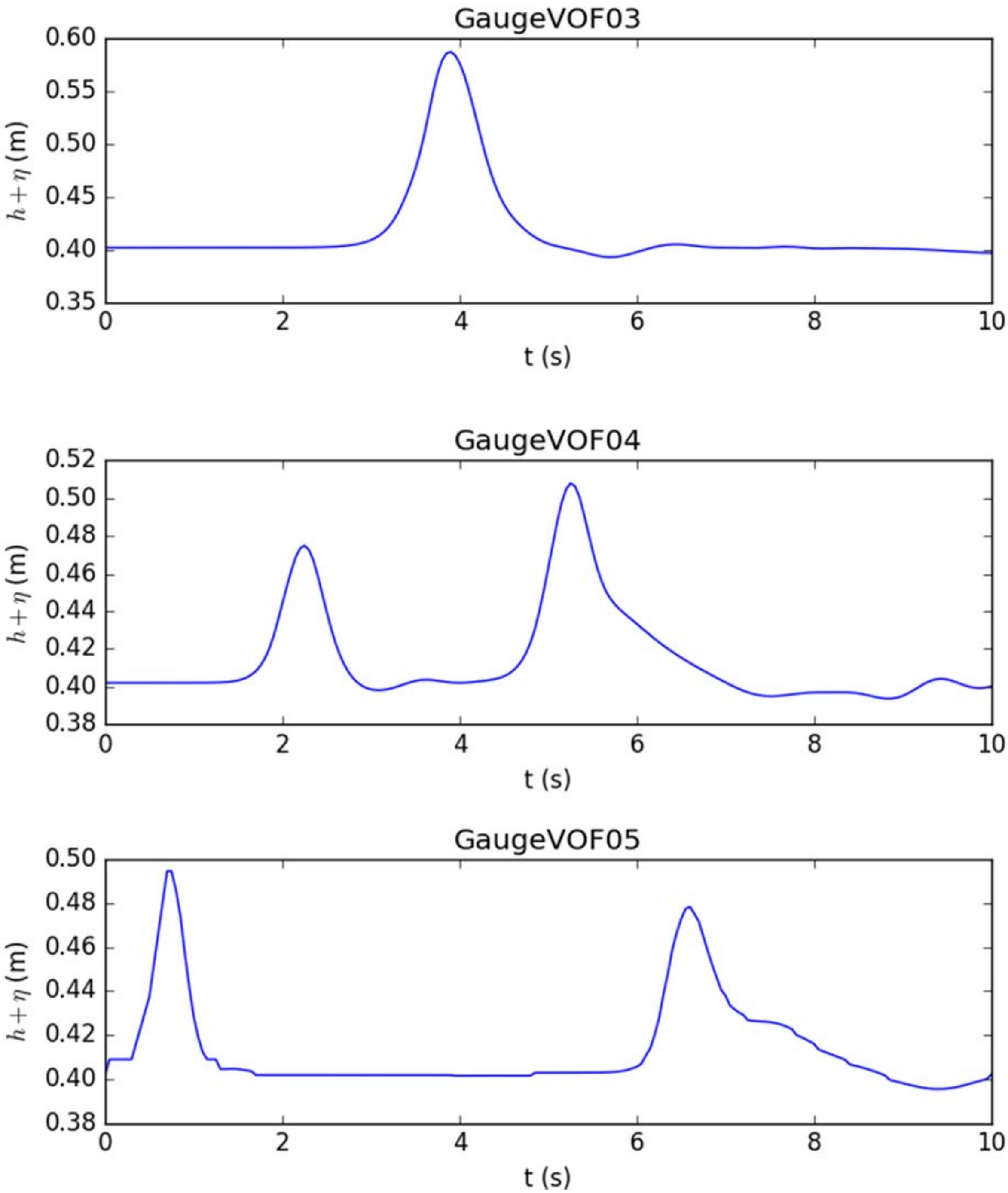

Figure 9. Interaction of two solitary waves in the opposite direction

2) Studies at ELTE

In the previous four semesters, I passed 120 credits. There are different topics related to my project was taken. In the first year, the main courses were “Solitons and Instantons” and “non-equilibrium statistical physics.” Participating in these courses, I have the opportunity to observe a wide variety of study methods in my research area. Besides, other courses allowed me to improve my abilities in programming. Python programming, data mining, and data visualization techniques were introduced in these courses. Doing projects is the second key factor that should be considered. They cover state-of-the-art physics and computational methods, particularly in Oceanography, Data Science, and Network Science.

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