Doctoral School of Physics - Eötvös Loránd University (ELTE)

Semester report

By **Arash Ghahraman** (arash.ghgood@gmail.com)

Doctoral School of Physics – ELTE

Supervisors: Dr. Gyula Bene

Ph.D. Thesis topic:

Study of solitons and applications with analytical and numerical methods

Personal circumstances:

In this semester, I continue my studies by taking courses entitled "*Solitons and Instantons II*", "*Data mining and machine learning*" and "*Data Science computer lab*" as well as guided research under the supervision of Dr. Bene. Participating in these courses, I have the opportunity to observe a wide variety of study methods in my research area. Particularly, in the "Solitons and Instantons II," I learned more about solitons and their application in a wide range of physics. In addition, I acquired skill in mathematical techniques. The other courses gave me the opportunity to improve my abilities on programming. Python programming and data mining techniques were introduced in these courses. Doing projects is the second key factor that should be considered. They cover state-of-the-art physics, particularly in Oceanography and Network Science.

Description of research work carried out in current semester:

To describe the nonlinear wave phenomenon remarkably in the ocean, various evolution equations have been proposed and studied. In shallow water, the Boussinesq equations for two-dimensional waves, the Korteweg-de Vries (KdV) equation for uni-directional waves, the Kadomtsev-Petviashvili (KP) equation for weakly two-dimensional waves and their solitary wave solutions are very well-known and of interests to all disciplines [1]. Many studies have attempted to assess the significance of dissipation. The viscous effects are a problematic situation as there is always a degree of ambiguity surrounding the concept of dissipation. The effect of viscosity on free oscillatory waves on deep water was studied by Boussinesq and Lamb, among others. Basset also worked on viscous damping of water waves [1, 2].

The consciousness of the effect of viscous dissipation is required in the assessment of the various process of wave generation [3]. Followed by previous studies we consider linear surface waves on a viscous, incompressible fluid of finite depth h. The coordinates x and y are horizontal, z is vertical. The origin lies at the undisturbed fluid surface. Suppose that the flow corresponding to the surface wave does not depend on y. We start with the Ansatz

$$u(x,z,t) = f'(z)e^{i(kx-\omega t)} \quad (1)$$

$$w(x,z,t) = -ik\, f(z)e^{i(kx-\omega t)} \quad (2)$$

for the non-vanishing velocity components. Note that ω is in general complex, its imaginary part describing the damping and the Ansatz automatically satisfies the incompressibility condition.

The linearized Navier-Stokes equation may be written as

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\,\Delta\mathbf{v} = \nabla\left(\frac{-1}{\rho}P(x,z,t) - gz\right) \quad (3)$$

So

$$\begin{aligned}
\frac{\partial u}{\partial t} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) &= e^{i(kx-\omega t)}\left[\left(k^2\nu - i\omega\right)f' - \nu f'''\right] = \frac{\partial}{\partial x}\left(-\frac{1}{\rho}P - gz\right) \\
\frac{\partial w}{\partial t} - \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) &= ik\,e^{i(kx-\omega t)}\left[\left(-k^2\nu + i\omega\right)f + \nu f''\right] = \frac{\partial}{\partial z}\left(-\frac{1}{\rho}P - gz\right)
\end{aligned} \quad (4)$$

Which leads to a differential equation for $f(z)$

$$f'''' + \left(i\frac{\omega}{\nu} - 2k^2\right)f'' - \left(i\frac{\omega}{\nu} - k^2\right)k^2 f = 0 \quad (5)$$

The general solution for f is

$$f = a_1\cosh\left(k(z+h)\right) + a_2\sinh\left(k(z+h)\right) + b_1\cosh\left(\kappa(z+h)\right) + b_2\sinh\left(\kappa(z+h)\right) \quad (6)$$

and $\kappa = \pm\sqrt{k^2 - i\dfrac{\omega}{\nu}}$. The linearized boundary conditions at the surface are

$$\frac{\partial \eta}{\partial t} = w(x,0,t) \quad (7)$$

$$\rho\nu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 \qquad at\ \ z = 0 \quad (8)$$

$$p - 2\rho\nu\frac{\partial w}{\partial z} = p_o \qquad at\ \ z = 0 \quad (9)$$

The free surface $z = \eta(x,t)$ must be found as part of solution. By defining the following variables

$$\begin{aligned}
K &= kh \\
Q &= \kappa h = \sqrt{1 - i\frac{\omega}{\nu k^2}}\ kh
\end{aligned} \quad (10)$$

$$\varepsilon = \left(\frac{4\nu^2 k^3}{g}\right)^{\frac{1}{4}} \quad (11)$$

we can rewrite Q in terms of K, ε and $S = \frac{\omega}{\sqrt{gk}}$.

$$2K^2 S = i\varepsilon^2 \left(Q^2 - K^2\right) \quad (12)$$

It is clear that in liquids of vanishing viscosity, as $\varepsilon \to 0$, S and K assume their inviscid significance and remain finite. Consequently, from (12), as $\varepsilon \to 0$ then $Q \to \infty$. Therefore, the kinematic boundary condition (7) accompanied with dynamic boundary conditions (8) and (9), at small viscosity limit, provide the dispersion relation

$$-4\left[\frac{Q}{K}\tanh(K) - 1\right] = \varepsilon^4\left[\left(\frac{Q}{K}\right)^5 + 2\left(\frac{Q}{K}\right)^3 + 5\frac{Q}{K} - \tanh(K)\left(\left(\frac{Q}{K}\right)^4 + 6\left(\frac{Q}{K}\right)^2 + 1\right)\right] \quad (13)$$

Our results fully agree with deep water solutions and particularly they absolutely lead to Zakharov's dispersion relation for small viscosity. The most significant benefit of (13) is that one can compute appropriate relation for finite depth and study shallow water $(K \to 0)$

In conclusion, this approach satisfies deep water limit and lets us study finite depth cases. The next step is to consider nonlinearity of the Navier-Stokes equation to next to the leading order. This would generalize the KdV equation for the viscous case. Furthermore, we would seek and compare theoretical results with OpenFoam which was designed last semester.

References

[1] F. Dias, A. I. Dyachenko and V. Zakharov, "Theory of weakly damped free-surface flow: a new formulation based on potential flow solutions," *Physics Letter A,* vol. 372, pp. 1297-1302, 2008.

[2] A. Latifi, M. A. Manna, P. Montalvo and M. Ruivo, "linear and weakly nonlinear models of wind generated surface waves in finite depth," *Journal of Applied Fluid Mechanics,* vol. 10, no. 6, pp. 1829-1843, 2017.

[3] T. H. C. HERBERS, S. ELGAR, N. A. SARAP and R. T. GUZA, "Nonlinear Dispersion of Surface Gravity Waves in Shallow Water," *J O U R N A L O F P H Y S I C A L O C E A N O G R A P H Y,* vol. 32, pp. 1181-1193, 2002.