Semester report 4

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Ph.D. Thesis title: Investigation of topological superconductors in the presence of interactions

1 Introduction

Key requirements for quantum computers include qubits with long decoherence times and a universal set of robust quantum gates [1]. Topological quantum computers address these through encoding in non-local quasiparticles [2,3]. Majorana fermions [4] and parafermions [5], which are excitations in topological superconductors [6, 7], fulfill these roles. Parafermions, classified by a $\mathbb{Z}_d$ index (with $d = 2$ for Majorana fermions), exhibit non-Abelian exchange statistics, enabling topologically protected quantum gates. Braiding $d > 2$ parafermion zero modes implements a broader set of quantum operations than Majorana zero modes [8].

Realizing parafermions typically relies on electron-electron interactions [7]. Some proposals require combining superconductivity with fractional quantum Hall edge modes, which is difficult due to conflicting magnetic field requirements [9, 10]. Recent theories suggest alternative platforms for $\mathbb{Z}_4$ parafermion zero modes using quantum spin Hall (QSH) insulators coupled to superconductors, avoiding the need for strong magnetic fields [11, 12]. In these setups, interactions create gaps in edge states while maintaining time-reversal symmetry, and parafermions emerge at interfaces between gapped regions and superconductors. These $\mathbb{Z}_4$ parafermions show a fourfold degenerate ground state and an 8π-periodic fractional Josephson effect.

Previous studies on parafermionic phases mostly utilize the bosonization technique, overlooking high-energy or lattice-scale effects [13], however, lattice models offer insights into the diverse landscape of parafermionic excitations. Density matrix renormalization group (DMRG) methods have been used to explore dynamical excitations in parafermionic chains [14]. Recent research thoroughly investigates the algebraic mapping between $\mathbb{Z}_4$ parafermions and spinful fermions [15]. Additionally, fermionized parafermion chains have been studied, revealing exotic electronic lattice models with unique interactions and hopping behaviors [16, 17].

2 Work carried out in the previous three semesters

In my first three semesters, I constructed a simple lattice model for interacting spinful electrons with parafermionic zero energy modes. By density matrix renormalization group calculations, I identified a broad range of parameters, with well-localized zero modes, whose parafermionic nature is substantiated by their unique 8π periodic Josephson spectrum. In this section, I introduce a few major points of the work. For details, see the article on arXiv [18].

2.1 The model

In the following, I introduce a ladder Hamiltonian acting on spinful electrons which captures the essential properties of the edge states of two-dimensional topological insulators without explicitly treating the insulating bulk. In the proposed model, each electronic site has local spin degrees of freedom denoted by $\uparrow$ and $\downarrow$, while the left and right leg of the ladder is referred to simply by the labels L and R respectively. The Hamiltonian of the system can be decomposed into three distinct parts:

$$H = H_{\mathrm{kin}} + H_{\mathrm{sc}} + H_{\mathrm{int}} \, . \tag{1}$$

The first term describes the kinetic contributions, capturing propagation along the legs and hopping across the rungs of the ladder,

$$H_{\text{kin}} = \sum_m c_m^\dagger \left(-\mu_m s_0 \otimes \zeta_0 + t s_0 \otimes \zeta_x\right) c_m \\ - \frac{t}{2} \sum_m c_{m+1}^\dagger \left(i s_z \otimes \zeta_z + s_0 \otimes \zeta_x\right) c_m + \text{h.c.} . \tag{2}$$

Here, s_α and ζ_α are Pauli matrices acting on the spin and leg degrees of freedom respectively, and $c_m^\dagger = \left(c_{m,\mathrm{L},\uparrow}^\dagger, c_{m,\mathrm{R},\uparrow}^\dagger, c_{m,\mathrm{L},\downarrow}^\dagger, c_{m,\mathrm{R},\downarrow}^\dagger\right)$ where $c_{m,\zeta,s}^\dagger$ denotes the creation operator of an electron with spin projection $s \in \{\uparrow, \downarrow\}$ on-site m of leg $\zeta \in \{\mathrm{L}, \mathrm{R}\}$. t serves as an overall energy scale for the system, while μ_m is a site-dependent potential.

For low energies, the kinetic term (2) describes the propagation of helical particles. Crucially no terms are mixing the two legs, thus for low-energy helical particles, the two legs are decoupled just as they would be for two spatially separated edges of a large two-dimensional topological insulator.

The second term in the Hamiltonian (1) describes proximity to an s-wave superconductor,

$$H_{\text{sc}} = \sum_{m,\zeta} \Delta_{m,\zeta} \left[c_{m,\zeta,\uparrow}^\dagger c_{m,\zeta,\downarrow}^\dagger + \text{h.c.}\right], \tag{3}$$

with a site and leg-dependent pair potential $\Delta_{m,\zeta}$.

The last term, H_{int}, describes a short ranged microscopic interaction

$$H_{\text{int}} = \sum_{m,\zeta} V_{m,\zeta} \left[c_{m,\zeta,\uparrow}^\dagger c_{m,\zeta,\downarrow} c_{m+1,\zeta,\uparrow}^\dagger c_{m+1,\zeta,\downarrow} + \text{h.c.}\right]. \tag{4}$$

2.2 Excitation spectrum

The phase with moderate interaction strength is endowed with one key indicator of the presence of $\mathbb{Z}_4$ parafermions, in the form of zero energy excitation. In Fig. 1, the low energy many-body excitation spectrum of the model is shown for $\Delta'/t = \mu/t = 0$ as the function of the interaction strength V. For weak interaction, up to around $V/t = 1.5$, the system has a well-defined ground state with even fermion parity, and the first excited state is a doubly degenerate odd state. In the thermodynamic limit, the spectrum of this phase shows metallic characteristics with a vanishing excitation gap. For intermediate interaction strengths, independent of the size of the system, a phase with a fourfold degenerate ground state emerges with a considerable gap in the excitation spectrum. At around $V/t = 3$ a second phase transition is observed. For stronger interaction strengths, the degeneracy is lifted and the gap increases linearly with V.

2.3 Josephson spectrum

The Josephson spectrum, shown in Fig. 2, indicates the presence of parafermionic zero modes in the considered model. The evolution of the energy of the localized four modes as the function of phase bias φ in between two superconducting terminals can be used to characterize anyonic excitations [19]. In particular, time-reversal invariant Majorana modes show a 4π periodic modulation [20, 21], while $\mathbb{Z}_4$ parafermions exhibit an 8π periodicity [7].

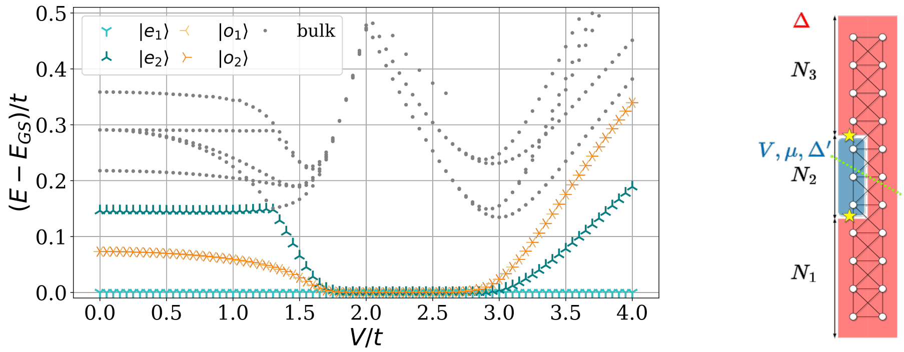

Figure 1: Left side: excitation spectrum of the considered model as the function of the interaction strength V. The two lowest energy even and odd parity states are marked by $|e_i\rangle$ and $|o_i\rangle$ with $i \in \{1, 2\}$. $N_i = 20$ and $\mu/t = 0$ in the configuration depicted on the right side.

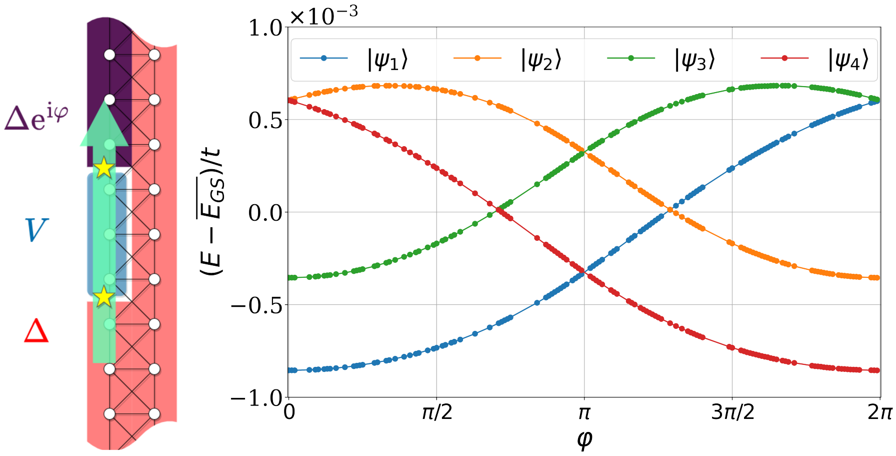

Figure 2: Left side: Schematic representation of a phase bias induced Josephson current crossing two parafermionic zero modes. Right side: Josephson spectrum (right panel) of a junction with $N = 8$ sites and interaction strength $V/t = 2.2$.

3 Work carried out in current semester

3.1 Adaptive phase exploration

One of the difficulties of creating a phase diagram is deciding the points for which the entanglement entropy is evaluated.

Up until this semester, I used two methods: partitioning the examined domain into equal-sized rectangles and picking random points uniformly from the domain. Out of these two, the second method was slightly better, because it yielded a "first-impression view" of the phases faster. However, the problem with both of these methods is that a high percentage of the points are "wasted" in the sense that they are placed in the "flat" parts of the phase diagram instead of the more interesting phase edge regions.

An interesting solution for this problem is offered by Adaptive [22], which, to put it simply, creates a triangular grid of the already existing points, assigns a loss value to each triangle, and picks triangles to detail randomly, with higher picking probability for higher loss triangles.

This new method sounded promising, but I wasn't yet able to achieve significant improvement with it. I tried several different loss functions, both built-in and custom ones (using for example triangle sizes, edge value differences, variances, and so on), and the losses themselves looked good (i.e. they had high values where I wanted to see more points) on my older data, but they did not perform as expected on new systems. Because of this, I decided to put it back on the shelves and come back to it in the future.

3.2 Polishing the first article

My first article was not accepted by Physical Review Letters, and because of that, a part of my resources was delegated to polishing it. This work, among other tasks, included doing extra calculations, reacting to reviewer comments, and refactoring the document.

3.3 Setbacks

My workstation, acquired through public procurement, broke at the start of March. Since it was used for most of my non-HPC requiring calculations, data post-processing, and data and code storing, I experienced a major setback. As of the beginning of June, the workstation remains under repair with no definitive timeline for its return. Additionally, I had to redo many calculations from the beginning due to this setback, causing further delays in my work.

4 Publications

My first paper, titled "A simple electronic ladder model harboring $\mathbb{Z}_4$ parafermions", is on the brink of submission to Physical Review B. It is already available on arXiv [18].

5 Studies in current semester

I attended one class in the current semester: "Green függvényes technika a nanofizikában" (subject code: "FIZ/1,3/068E")

6 Conferences in current semester

In the current semester, I was going to attend "Workshop: Recent progress on tensor network methods" at Technical University of Munich, but could not do so due to unforeseen health-related issues.

7 Teaching activity in current semester

In the current semester, I participated as a lecturer (for 1 class/week, i.e. 2 hours/week) in the practice class "A fizika numerikus módszerei 1" (with subject code "fiznum1f19la"), which is a course about useful python tools in physics for second-semester Physics BSc students.

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